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## Half-transitivity of some metacirculants

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### Abstract

In 1991, Alspach, Marušič, and Nowitz proved that there are infinitely many  $\frac{1}{2}$ -transitive graphs of degree 4. Their graphs were found among metacirculants  $M(\alpha; m, n)$ , which have vertex set  $\{v_j^i: i \in Z_m, j \in Z_n\}$  and edge set  $\{v_j^i v_{j+\delta\alpha}^{i+1}: i \in Z_m, j \in Z_n, \delta \in \{-1, 1\}\}$  with the additional condition that  $\alpha \in Z_n^*$  has order  $m$  or  $2m$ . Examining only the cases when both  $m$  and  $n$  are odd, they showed that the graphs  $M(\alpha; 3, n)$  are  $\frac{1}{2}$ -transitive when  $n \geq 9$  and gave a sufficient condition for  $M(\alpha; m, n)$  to be  $\frac{1}{2}$ -transitive when  $m$  is composite and  $n$  is prime. In this paper, we give a simple generalization of this condition. We also show that the graphs  $M(\alpha; 2, n)$  are arc-transitive. Then we examine the graphs  $M(\alpha; 4, n)$ . We prove that they are arc-transitive when the order of  $\alpha$  is 4 with  $\alpha^2 \equiv -1 \pmod{n}$  and  $\frac{1}{2}$ -transitive when either the order of  $\alpha$  is 8 or the order of  $\alpha$  is 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$  and  $n$  is not a multiple of 4. © 1998 Elsevier Science B.V. All rights reserved

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### 1. Introduction

An *arc* is an ordered pair of adjacent vertices in a graph. A graph is *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group acts transitively on the vertex set, edge set, and arc set, respectively. A graph is said to be  $\frac{1}{2}$ -transitive if it is vertex- and edge-transitive but not arc-transitive. In 1966, Tutte [7] showed that every vertex- and edge-transitive graph of odd degree is also arc-transitive. Four years later Bouwer [4] showed that this does not extend to vertex- and edge-transitive graphs of even degree by constructing one  $\frac{1}{2}$ -transitive graph of degree  $N$  for each even  $N$  greater than 2. The smallest of Bouwer's graphs has order 54 and degree 4. A smaller  $\frac{1}{2}$ -transitive graph of degree 4 was found by Holt [5] in 1981. Alspach et al. [2] proved later that this graph of order 27 is the smallest  $\frac{1}{2}$ -transitive graph, and Xu [8] showed its uniqueness. In addition, Alspach et al. [2] found the first infinite family of  $\frac{1}{2}$ -transitive graphs of degree 4. These graphs are all metacirculants  $M(\alpha; m, n)$ . In this paper, we extend the results of Alspach et al. [2] regarding  $\frac{1}{2}$ -transitivity of metacirculants  $M(\alpha; m, n)$ .

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## 2. Definitions, preliminary results, and a corollary

A *metacirculant*  $G = G(m, n, \alpha, S_0, S_1, \dots, S_\mu)$  is a graph with vertex set  $\{v_j^i: i \in Z_m, j \in Z_n\}$  and edge set  $\{v_j^i v_{j+h}^{i+r}: 0 \leq r \leq \mu \text{ and } h \in \alpha^i S_r\}$ , where  $m \geq 1$ ,  $n \geq 2$ ,  $\alpha \in Z_n^*$ ,  $\mu = \lfloor m/2 \rfloor$ , and  $S_0, S_1, \dots, S_\mu$  are subsets of  $Z_n$  satisfying the following conditions:  $0 \notin S_0 = -S_0$ ,  $\alpha^m S_r = S_r$  for  $0 \leq r \leq \mu$ , and  $\alpha^\mu S_\mu = -S_\mu$  if  $m$  is even.

A graph  $M(\alpha; m, n)$  is the metacirculant  $G(m, n, \alpha, \emptyset, \{-1, 1\}, \emptyset, \dots, \emptyset)$ , where  $m \geq 2$ ,  $n \geq 5$ , and  $\alpha \in Z_n^*$  has order  $m$  or  $2m$ . That is,  $M(\alpha; m, n)$  is the graph with vertex set  $V = \{v_j^i: i \in Z_m, j \in Z_n\}$  and edge set  $E = \{v_j^i v_{j+\delta\alpha^i}^{i+1}: i \in Z_m, j \in Z_n, \delta \in \{-1, 1\}\}$ . Notice that, if the order of  $\alpha$  is  $2m$ ,  $\alpha^m \equiv -1 \pmod{n}$ . The sets  $V^i = \{v_j^i: j \in Z_n\}$ ,  $i = 0, \dots, m$ , are called the *blocks* of  $M(\alpha; m, n)$  although they need not be blocks of imprimitivity of its automorphism group. (For a more detailed discussion of metacirculants and the graphs  $M(\alpha; m, n)$  the reader is referred to [1, 2].)

The graphs  $M(\alpha; m, n)$  are vertex- and edge-transitive. Notice that the mappings  $\rho, \tau$ , and  $\pi$  defined by  $\rho(v_j^i) = v_{j+1}^i$ ,  $\tau(v_j^i) = v_{\alpha j}^{i+1}$ , and  $\pi(v_j^i) = v_{-j}^i$ , respectively, are all automorphisms of the graph  $M(\alpha; m, n)$ . In fact,  $\rho, \tau$ , and  $\pi$  generate a  $\frac{1}{2}$ -transitive subgroup  $H$  of the automorphism group of  $M(\alpha; m, n)$ .

Our aim is to determine which of the graphs  $M(\alpha; m, n)$  are arc-transitive and which are  $\frac{1}{2}$ -transitive.

**Theorem 2.1.** *Every graph  $M(\alpha; 2, n)$  is arc-transitive.*

**Proof.** The mapping  $\sigma$  defined by  $\sigma(v_j^i) = v_{-j+1}^{i+1}$  is an automorphism of the graph that reverses the edge  $v_0^0 v_1^1$ .  $\square$

For the rest of the paper we may assume that  $m$  is at least 3.

A cycle  $C$  of  $M(\alpha; m, n)$  of length at least  $m$  is said to be *coiled* if every subpath of  $C$  having  $m$  vertices intersects each of  $V^0, V^1, \dots, V^{m-1}$ . Clearly, a coiled cycle must have length a multiple of  $m$ . The *coiled girth* of  $M(\alpha; m, n)$  is the length of a shortest coiled cycle in  $M(\alpha; m, n)$ . It is not difficult to see that the coiled girth of  $M(\alpha; m, n)$  is either  $m$  or  $2m$ .

**Definition 2.2.** Let  $a \in \{m, 2m\}$  be the order of  $\alpha$  in  $Z_n^*$ . Let  $P = v_{j_0}^{i_0} v_{j_1}^{i_1} \dots v_{j_k}^{i_k}$  be a path in  $M(\alpha; m, n)$  and let  $\Delta_l \in \{\pm \alpha^i: i = 0, 1, \dots, a-1\}$  for  $l = 0, 1, 2, \dots, k-1$ . If  $\Delta_l = j_{l+1} - j_l$  for  $l = 0, 1, \dots, k-1$ , then

$$\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$$

is called a *jump sequence* of the path  $P$ . Notice that if  $\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$  is a jump sequence of a given path, then so is  $\langle -\Delta_{k-1}, \dots, -\Delta_1, -\Delta_0 \rangle$ . In addition, if  $v_{j_0}^{i_0} = v_{j_k}^{i_k}$ , that is, if  $P$  is a cycle, then cyclically permuting the entries in  $\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$  yields another jump sequence for  $P$ . We shall not distinguish between jump sequences of the same path or cycle.

If  $\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$  is the jump sequence of a cycle, then

$$\Delta_0 + \Delta_1 + \dots + \Delta_{k-1} \equiv 0 \pmod{n}$$

must hold. This is the *congruence equality* associated with the jump sequence  $\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$ .

If  $\langle \Delta_0, \Delta_1, \dots, \Delta_{k-1} \rangle$  is the jump sequence for the path  $P$ , then for any  $s \in \{0, 1, \dots, a-1\}$  and any  $\delta \in \{-1, 1\}$ ,

$$\langle \delta \alpha^s \Delta_0, \delta \alpha^s \Delta_1, \dots, \delta \alpha^s \Delta_{k-1} \rangle$$

is called a *type sequence* of the path  $P$ . Two type sequences are called *equivalent* if one can be obtained from the other by cyclically permuting the entries (for cycles only), reversing the order and/or the signs of the entries, and/or multiplying all entries by the same power of  $\alpha$ . In other words, all type sequences of the same path or cycle are equivalent. Notice that, for cycles, equivalent type sequences give equivalent congruence equalities in the sense that they differ only by a factor of  $\pm \alpha^i$ .

The sequence  $\langle i_0, i_1, \dots, i_{k-1}, i_k \rangle$  or  $\langle i_0, i_1, \dots, i_{k-1} \rangle$  will be referred to as a *block sequence* of the path or cycle, respectively,  $P = v_{j_0}^{i_0} v_{j_1}^{i_1} \dots v_{j_{k-1}}^{i_{k-1}} v_{j_k}^{i_k}$ . Notice that for two block sequences of the same cycle, one can be obtained from the other by cyclically permuting the entries and/or reversing their order. We shall not distinguish between block sequences of the same cycle.

Let  $M = M(\alpha; m, n)$  have coiled girth  $m$ . Fix a vertex  $v_j^i$  of  $M$ . The coiled  $m$ -cycles that contain the vertex  $v_j^i$  occur in pairs: the jump sequence of one coiled  $m$ -cycle of the pair is obtained from the jump sequence of the other cycle of the pair by reversing the signs of all entries. The graph  $M(\alpha; m, n)$  of coiled girth  $m$  is said to be *tightly coiled* if every vertex is incident with exactly two coiled  $m$ -cycles; otherwise (that is, if every vertex is incident with at least four coiled  $m$ -cycles), the graph is *loosely coiled*. The following result was proved by Alspach et al.

**Theorem 2.3** (Alspach et al. [2]). *Let  $m$  and  $n$  be odd and let  $M = M(\alpha; m, n)$  have coiled girth  $m$  and be loosely coiled. Then  $M$  is  $\frac{1}{2}$ -transitive.*

In general, it is not easy to determine for which values of the parameters  $\alpha$ ,  $m$ , and  $n$  the graph  $M(\alpha; m, n)$  has coiled girth  $m$  and is loosely coiled. In [2], the authors prove the following sufficient condition:  $n$  is prime and  $\alpha$  is a divisor of  $n-1$  whose order  $m$  is odd and composite. We extend this condition in the next corollary.

**Corollary 2.4.** *Let  $p$  be an odd prime such that  $p-1 = km'd$  where  $m' > 1$  and  $d > 1$  are both odd. Let  $n = p^s$  for some  $s \in \mathbb{N}$  and let  $\alpha \in Z_n^*$  have order  $m = m'dp^{s-1}$ . Then the graph  $M = M(\alpha; m, n)$  is  $\frac{1}{2}$ -transitive.*

**Proof.** Since  $n$  is an odd prime power, the group  $Z_n^*$  is cyclic (see [3]). Since  $m$  divides its order  $(p-1)p^{s-1}$ , there exists  $\alpha \in Z_n^*$  with order  $m$ .

Next we show that  $\alpha^d - 1$  cannot be a zero divisor in the ring  $Z_n$ . Assume the contrary. Then  $\alpha^{td} - 1$  is a zero divisor for all  $t \in \{1, 2, \dots, m'p^{s-1} - 1\}$ . Since the order of  $\alpha$  is  $m$ , the elements  $\alpha^{td} - 1, t = 1, 2, \dots, m'p^{s-1} - 1$ , are pairwise distinct. We would thus have  $m'p^{s-1} - 1 > p^{s-1} - 1$  zero divisors in  $Z_n$ , a contradiction.

Hence,  $\alpha^d - 1 \in Z_n^*$  and, consequently,  $\alpha - 1 \in Z_n^*$ . Therefore,  $\alpha^m \equiv 1 \pmod{n}$  implies

$$1 + \alpha^d + \alpha^{2d} + \dots + \alpha^{(m/d-1)d} \equiv 0 \pmod{n} \quad (1)$$

and

$$1 + \alpha + \alpha^2 + \dots + \alpha^{m-1} \equiv 0 \pmod{n}. \quad (2)$$

The last equation shows that  $\langle 1, \alpha, \alpha^2, \dots, \alpha^{m-1} \rangle$  is a jump sequence of a coiled  $m$ -cycle so that  $M$  has coiled girth  $m$ . From Eqs. (1) and (2) we obtain

$$\begin{aligned} -1 + \alpha + \dots + \alpha^{d-1} - \alpha^d + \alpha^{d+1} + \dots + \alpha^{2d-1} - \alpha^{2d} + \alpha^{2d+1} \\ + \dots + \alpha^{m-1} \equiv 0 \pmod{n}, \end{aligned}$$

which implies that  $M$  contains coiled  $m$ -cycles with the jump sequence

$$\langle -1, \alpha, \dots, \alpha^{d-1}, -\alpha^d, \alpha^{d+1}, \dots, \alpha^{2d-1}, -\alpha^{2d}, \alpha^{2d+1}, \dots, \alpha^{m-1} \rangle.$$

Hence,  $M$  is loosely coiled and thus,  $\frac{1}{2}$ -transitive by Theorem 2.3.  $\square$

### 3. Metacirculants $M(\alpha; 4, n)$

Alspach et al. [2] also showed that every graph  $M(\alpha; 3, n)$  is  $\frac{1}{2}$ -transitive whenever  $n \geq 9$  is odd, and recently Marušič [6] characterized the  $\frac{1}{2}$ -transitive graphs  $M(\alpha; m, n)$  with  $m \geq 5$  and  $n$  odd. Since the two-block case is settled (Theorem 2.1), our next point of interest is the graphs  $M(\alpha; 4, n)$ . The purpose of this section is to prove the following theorem.

**Theorem 3.1.** *A graph  $M(\alpha; 4, n)$  is arc-transitive if  $\alpha^2 \equiv -1 \pmod{n}$ , and  $\frac{1}{2}$ -transitive if either  $\alpha^4 \equiv -1 \pmod{n}$  or the order of  $\alpha$  in  $Z_n^*$  is 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$  and  $n$  is not a multiple of 4.*

Notice that the only case when neither arc-transitivity nor  $\frac{1}{2}$ -transitivity has been established is when the order of  $\alpha$  is 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$  and  $n$  is a multiple of 4. We prove the first part of Theorem 3.1 in the next lemma.

**Lemma 3.2.** *The graph  $M = M(\alpha; 4, n)$  is arc-transitive if  $\alpha^2 \equiv -1 \pmod{n}$ .*

**Proof.** Define  $\sigma(v_j^i) = v_{-j+1}^{-i+1}$ . Since  $\alpha^2 \equiv -1 \pmod{n}$ , for any  $\delta \in \{-1, 1\}$ ,

$$\sigma(v_j^i v_{j+\delta\alpha^i}^{i+1}) = v_{-j+1}^{-i+1} v_{-j+1-\delta\alpha^{-i}}^{-i} \quad \text{if } i \in \{0, 2\},$$

and

$$\sigma(v_j^i v_{j+\delta\alpha^i}^{i+1}) = v_{-j+1}^{-i+1} v_{-j+1+\delta\alpha^{-i}}^{-i} \quad \text{if } i \in \{1, 3\},$$

so that  $\sigma$  is an automorphism of  $M$ . Since  $\sigma$  interchanges the endpoints of the edge  $v_0^0 v_1^1$ ,  $M$  is arc-transitive.  $\square$

**Corollary 3.3.** *If  $n$  is an odd prime power and  $\alpha \in Z_n^*$  has order 4, then  $M(\alpha; 4, n)$  is arc-transitive.*

Before we proceed to give a brief outline of the rest of the proof, consider the action of the  $\frac{1}{2}$ -transitive subgroup  $H$  of the automorphism group of  $M = M(\alpha; 4, n)$  (see Section 2) on the 2-paths of  $M$ . It is not difficult to see that  $H$  has four orbits on the 2-paths, one corresponding to each of the type sequences  $\langle \alpha^3, 1 \rangle$  and  $\langle -\alpha^3, 1 \rangle$ , one corresponding to the type sequence  $\langle 1, 1 \rangle$  with the block sequence of the form  $\langle\langle i+1, i, i+1 \rangle\rangle$ , and one corresponding to the type sequence  $\langle 1, 1 \rangle$  with the block sequence of the form  $\langle\langle i, i+1, i \rangle\rangle$ . Notice, however, that the block sequence of every cycle contains as many subsequences of the form  $\langle\langle i+1, i, i+1 \rangle\rangle$  as those of the form  $\langle\langle i, i+1, i \rangle\rangle$ . In fact, these two kinds of subsequences alternate around the cycle. It follows then that every 2-path with a given type sequence lies in the same number of cycles with a given type sequence.

**Definition 3.4.** The triple  $(k_1, k_2, k_3)$  is called the *2-path code* of the graph  $M(\alpha; 4, n)$  if every 2-path with the type sequence  $\langle 1, 1 \rangle$  lies in exactly  $k_1$  8-cycles, every 2-path with the type sequence  $\langle \alpha^3, 1 \rangle$  lies in exactly  $k_2$  8-cycles, and every 2-path with the type sequence  $\langle -\alpha^3, 1 \rangle$  lies in exactly  $k_3$  8-cycles in  $M(\alpha; 4, n)$ .

Similarly,  $(k_1, k_2, k_3)$  is called the *2-path code* of a given type sequence of an 8-cycle if every 2-path with the type sequence  $\langle 1, 1 \rangle$  lies in exactly  $k_1$  8-cycles, every 2-path with the type sequence  $\langle \alpha^3, 1 \rangle$  lies in exactly  $k_2$  8-cycles, and every 2-path with the type sequence  $\langle -\alpha^3, 1 \rangle$  lies in exactly  $k_3$  8-cycles with the given type sequence.

The approach we use in proving the rest of Theorem 3.1 is classification of all 8-cycles in the graph. First, we determine all possible block sequences of an 8-cycle. For each block sequence, all possible type sequences are found, and for each type sequence the corresponding congruence equality is examined by a method we call the squaring elimination. In most cases, this method answers the question whether or not an 8-cycle with a given type sequence occurs in the graph. Once an 8-cycle is confirmed, its 2-path code is determined, and from the 2-path codes of all 8-cycles occurring in the graph we get the 2-path code of the graph. In case of a ‘nice’ 2-path code, Proposition 3.14 shows that the graph is  $\frac{1}{2}$ -transitive. In the four exceptional cases a modification of Proposition 3.14 must be used (Lemma 3.15). This approach was inspired by Bouwer’s proof in [4], and is outlined in Fig. 1. It can also be used in the three-block case to provide a slightly shorter alternative to the proof by Alspach et al. [2].

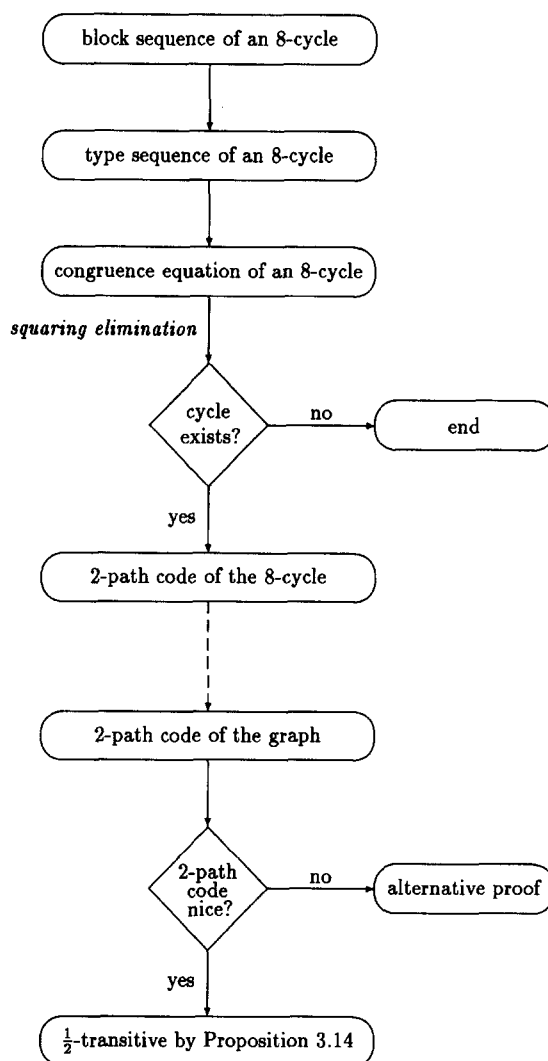


Fig. 1. Outline of the proof of Theorem 3.1.

We begin with a couple of lemmas concerning general graphs  $M(\alpha; m, n)$ . The first lemma implies that in a graph  $M(\alpha; 4, n)$ ,  $n$  is not a multiple of 4 if the order of  $\alpha$  is 8, and the second allows us to restrict ourselves to graphs  $M(\alpha; 4, n)$  with  $n$  odd.

**Lemma 3.5.** *Let  $n \equiv 0 \pmod{4}$  and  $\alpha \in Z_n^*$ . If  $\alpha^m \equiv -1 \pmod{n}$ , then  $m$  is odd and  $\alpha \equiv 3 \pmod{4}$ . If  $\alpha^m \equiv 1 \pmod{n}$ , then  $m$  is even or  $\alpha \equiv 1 \pmod{4}$ .*

**Proof.** Since  $\alpha \in Z_n^*$  and  $n$  is even,  $\alpha$  must be odd. Let  $\alpha = 2k + 1$  for some  $k \in Z_n$ . If  $\alpha^m \equiv -1 \pmod{n}$ , then  $\alpha^m + 1 \equiv 0 \pmod{4}$ . Hence,  $0 \equiv \alpha^m + 1 \equiv 2(mk + 1) \pmod{4}$  so that  $m$  and  $k$  must both be odd. Thus,  $\alpha \equiv 3 \pmod{4}$ .

If  $\alpha^m \equiv 1 \pmod{n}$ , then  $\alpha^m - 1 \equiv 0 \pmod{4}$ . Hence,  $0 \equiv \alpha^m - 1 \equiv 2mk \pmod{4}$  so that either  $m$  is even or  $\alpha \equiv 1 \pmod{4}$  or both.

**Lemma 3.6.** *Let  $n$  be even. Then  $M(\alpha; m, n)$  is connected if and only if  $m$  is odd. If  $n \equiv 2 \pmod{4}$  and  $m$  is even, then  $M(\alpha; m, n)$  consists of two connected components which are both isomorphic to  $M(\alpha \bmod \frac{n}{2}; m, \frac{n}{2})$ .*

**Proof.** Since  $n$  is even,  $\alpha$  is odd and the subgraph  $M[V^i, V^{i+1}]$  induced by two adjacent blocks consists of two disjoint  $n$  cycles. One cycle alternates between vertices of  $V^i$  with odd subscripts and vertices of  $V^{i+1}$  with even subscripts, and the other cycle alternates between vertices of  $V^i$  with even subscripts and vertices of  $V^{i+1}$  with odd subscripts. It is now clear that  $M(\alpha; m, n)$  is connected if and only if  $m$  is odd.

Now let  $n \equiv 2 \pmod{4}$  and let  $m$  be even. Then we get exactly two components and  $\rho(v_j^i) = v_{j+1}^i$  is an isomorphism between them. Choose the component with vertex set  $V' = \{v_j^i: i \in Z_m, j \in Z_n, i \equiv j \pmod{2}\}$  and let  $V_k$  be the vertex set of the graph  $M(\alpha \bmod k; m, k)$ , where  $k = n/2$ . Define  $\sigma: V' \rightarrow V_k$  by  $\sigma(v_j^i) = v_{j \bmod k}^i$ . Since each of the sets  $\{0, 2, \dots, n-2\}$  and  $\{1, 3, \dots, n-1\}$  modulo  $k$  produces all of  $Z_k$ ,  $\sigma$  is a bijection. Since, in addition,  $l - j \equiv \alpha^i \pmod{n}$  implies  $(l \bmod k) - (j \bmod k) \equiv (\alpha \bmod k)^i \pmod{k}$  for any  $j, l \in Z_n$  and  $i \in Z_m$ ,  $\sigma$  is an isomorphism between the connected component of  $M(\alpha; m, n)$  with vertex set  $V'$  and  $M(\alpha \bmod k; m, k)$ .  $\square$

The next lemma, whose proof is left to the reader, allows us to consider the graphs  $M(\alpha; 4, n)$  with  $n$  fixed for only one multiplier  $\alpha$  whenever all possible multipliers are  $\alpha, -\alpha, \alpha^3$ , and  $-\alpha^3$ . In particular, if  $n$  is a prime, there is (up to isomorphism) at most one graph  $M(\alpha; 4, n)$  once the order of  $\alpha$  is specified.

**Lemma 3.7.** *The graphs  $M(\alpha; m, n)$ ,  $M(-\alpha; m, n)$ , and  $M(\alpha^{-1}; m, n)$  are isomorphic.*

We proceed to explore coiled 8-cycles in the graphs  $M(\alpha; 4, n)$ . Recall Definitions 2.2. In the proof of the next lemma we describe the squaring elimination method which will be used extensively in subsequent proofs.

**Lemma 3.8.** *If  $n$  is odd and  $\alpha^4 \equiv -1 \pmod{n}$ , then  $M(\alpha; 4, n)$  has coiled girth 8. Furthermore, every coiled 8-cycle has a type sequence*

$$\langle 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3, -1, -\delta_1 \alpha, -\delta_2 \alpha^2, -\delta_3 \alpha^3 \rangle,$$

where  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$ . All choices for the  $\delta_i$  are realizable in the graph.

**Proof.** Note that  $\varphi(n)$  must be a multiple of 8, where  $\varphi$  is the Euler  $\varphi$ -function.

First we show that  $M(\alpha; 4, n)$  has coiled girth 8. Contrarily, suppose that the graph contains a coiled 4-cycle with a type sequence  $\langle 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3 \rangle$  for some  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$ . Then

$$1 + \delta_1 \alpha + \delta_2 \alpha^2 + \delta_3 \alpha^3 \equiv 0 \pmod{n}. \quad (3)$$

Expressing

$$\delta_2 \alpha^2 \equiv -(1 + \delta_1 \alpha + \delta_3 \alpha^3) \pmod{n}$$

and then squaring both sides we obtain

$$-1 \equiv 1 + \alpha^2 - \alpha^2 + 2\delta_1 \alpha + 2\delta_3 \alpha^3 - 2\delta_1 \delta_3 \pmod{n}$$

or

$$(1 - \delta_1 \delta_3) + \delta_1 \alpha + \delta_3 \alpha^3 \equiv 0 \pmod{n} \quad (4)$$

since  $\gcd(2, n) = 1$ . From equalities (3) and (4) we can eliminate the terms with  $\alpha$  and  $\alpha^3$  simultaneously thus obtaining

$$\delta_1 \delta_3 + \delta_2 \alpha^2 \equiv 0 \pmod{n}$$

or

$$-\delta_2 \alpha^2 \equiv \delta_1 \delta_3 \pmod{n}.$$

Squaring both sides again yields  $-1 \equiv 1 \pmod{n}$  which in turn implies  $n = 2$ , a contradiction. Hence,  $M(\alpha; 4, n)$  has coiled girth 8. The above method will be called *squaring elimination*.

We now explore coiled 8-cycles. A coiled 8-cycle has a type sequence

$$\langle \delta'_0, \delta'_1 \alpha, \delta'_2 \alpha^2, \delta'_3 \alpha^3, \delta''_0, \delta''_1 \alpha, \delta''_2 \alpha^2, \delta''_3 \alpha^3 \rangle$$

for some  $\delta'_i, \delta''_i \in \{-1, 1\}$ ,  $i = 0, \dots, 3$ , such that

$$\delta'_0 + \delta'_1 \alpha + \delta'_2 \alpha^2 + \delta'_3 \alpha^3 + \delta''_0 + \delta''_1 \alpha + \delta''_2 \alpha^2 + \delta''_3 \alpha^3 \equiv 0 \pmod{n}.$$

This equation can be simplified to

$$\varepsilon_0 + \varepsilon_1 \alpha + \varepsilon_2 \alpha^2 + \varepsilon_3 \alpha^3 \equiv 0 \pmod{n}, \quad (5)$$

where  $\varepsilon_i = \frac{1}{2}(\delta'_i + \delta''_i) \in \{-1, 0, 1\}$ . We now apply the squaring elimination method, described in the first part of the proof, to (5) thus obtaining

$$-(\varepsilon_3^2 - \varepsilon_1^2 + 2\varepsilon_0 \varepsilon_2)^2 \equiv (\varepsilon_0^2 - \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_3)^2 \pmod{n}.$$

The only possible values for  $(\varepsilon_3^2 - \varepsilon_1^2 + 2\varepsilon_0 \varepsilon_2)^2$  and  $(\varepsilon_0^2 - \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_3)^2$  are 0, 1, 4, and 9. But whenever  $-\varepsilon \equiv \varepsilon' \pmod{n}$  for  $\varepsilon, \varepsilon' \in \{0, 1, 4, 9\}$  and both  $\varepsilon$  and  $\varepsilon'$  are non-zero,  $n \in \{3, 5, 9, 13\}$  is forced, which is a contradiction. Hence  $\varepsilon_3^2 - \varepsilon_1^2 + 2\varepsilon_0 \varepsilon_2 = \varepsilon_0^2 - \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_3 = 0$ , which implies  $\varepsilon_i = 0$  for all  $i = 0, \dots, 3$ . Therefore  $\delta''_i = -\delta'_i$  for  $i = 0, \dots, 3$ , and the lemma is proved.  $\square$

Next, we state a result similar to that of Lemma 3.8 for the case that the order of  $\alpha$  is 4. Notice that, by Lemma 3.2, we may assume that  $\alpha^2 \not\equiv -1 \pmod{n}$  and hence that  $n$  is not an odd prime power.



**Lemma 3.9.** *Let  $n$  be odd and let  $\alpha$  have order 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$ . If  $M = M(\alpha; 4, n)$  has coiled girth 4, then (replacing  $\alpha$  by  $-\alpha$  if necessary) every coiled 4-cycle in  $M$  has a type sequence*

$$\langle 1, \alpha, \alpha^2, \alpha^3 \rangle.$$

*Furthermore, every coiled 8-cycle has a type sequence of the form*

$$\langle 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3, -1, -\delta_1 \alpha, -\delta_2 \alpha^2, -\delta_3 \alpha^3 \rangle,$$

*where  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$ . If  $M$  has coiled girth 8, all choices for  $\delta_1, \delta_2$ , and  $\delta_3$  are realizable. If  $M$  has coiled girth 4, all choices for  $\delta_1, \delta_2$ , and  $\delta_3$  are realizable except for  $\delta_1 = \delta_2 = \delta_3 = 1$ .*

**Proof.** First assume that  $M$  has coiled girth 4. If  $\alpha^3 + \alpha^2 - \alpha - 1 \equiv 0 \pmod{n}$ , then  $\alpha^3 + \alpha^2 + \alpha + 1 \equiv 2(\alpha + 1) \pmod{n}$  so that  $0 \equiv \alpha^4 - 1 \equiv (\alpha - 1)(\alpha^3 + \alpha^2 + \alpha + 1) \equiv 2(\alpha - 1)(\alpha + 1) \equiv 2(\alpha^2 - 1) \pmod{n}$ , a contradiction. Hence, either  $\alpha^3 + \alpha^2 + \alpha + 1 \equiv 0 \pmod{n}$ ,  $\alpha^3 + \alpha^2 + \alpha - 1 \equiv 0 \pmod{n}$ , or  $\alpha^3 - \alpha^2 + \alpha - 1 \equiv 0 \pmod{n}$ . Since  $\alpha^4 \equiv 1 \pmod{n}$ , either  $\alpha^3 + \alpha^2 + \alpha + 1 \equiv 0 \pmod{n}$  or  $\alpha^3 + \alpha^2 + \alpha + 1$  is a zero divisor in  $Z_n$ . If  $\alpha^3 + \alpha^2 + \alpha + 1 \equiv 0 \pmod{n}$ , none of the other two congruences holds. Therefore, every coiled 4-cycle has a type sequence  $\langle 1, \alpha, \alpha^2, \alpha^3 \rangle$ .

Now, assume that  $\alpha^3 + \alpha^2 + \alpha + 1$  is a zero divisor in  $Z_n$ . Since 2 is not a zero divisor,  $\alpha^3 + \alpha^2 + \alpha - 1 \not\equiv 0 \pmod{n}$ . Hence  $\alpha^3 - \alpha^2 + \alpha - 1 \equiv 0 \pmod{n}$  is the only possibility. But then  $(-\alpha)^3 + (-\alpha)^2 + (-\alpha) + 1 \equiv 0 \pmod{n}$ . Therefore, since the graphs  $M(\alpha; 4, n)$  and  $M(-\alpha; 4, n)$  are isomorphic, we may assume that if  $M(\alpha; 4, n)$  has coiled girth 4, then  $\alpha^3 + \alpha^2 + \alpha + 1 \equiv 0 \pmod{n}$ .

Next we explore coiled 8-cycles. As in the proof of the previous lemma, a coiled 8-cycle is associated with the equation

$$\varepsilon_0 + \varepsilon_1 \alpha + \varepsilon_2 \alpha^2 + \varepsilon_3 \alpha^3 \equiv 0 \pmod{n}, \quad (6)$$

where  $\varepsilon_i \in \{-1, 0, 1\}$ . Applying the squaring elimination method described before and using  $(\alpha^2)^2 \equiv 1 \pmod{n}$ , (6) yields

$$(\varepsilon_1^2 + \varepsilon_3^2 - 2\varepsilon_0\varepsilon_2)\alpha^2 \equiv \varepsilon_0^2 + \varepsilon_2^2 - 2\varepsilon_1\varepsilon_3 \pmod{n}$$

and, squaring again,

$$(\varepsilon_1^2 + \varepsilon_3^2 - 2\varepsilon_0\varepsilon_2)^2 \equiv (\varepsilon_0^2 + \varepsilon_2^2 - 2\varepsilon_1\varepsilon_3)^2 \pmod{n}.$$

The only possible values for  $\varepsilon = \varepsilon_1^2 + \varepsilon_3^2 - 2\varepsilon_0\varepsilon_2$  and  $\varepsilon' = \varepsilon_0^2 + \varepsilon_2^2 - 2\varepsilon_1\varepsilon_3$  are 0,  $\pm 1$ ,  $\pm 2$ , 3 and 4. Thus, if  $\varepsilon^2 \equiv \varepsilon'^2 \pmod{n}$  and  $\varepsilon^2 \neq \varepsilon'^2$ ,  $n = 15$  is forced. This implies  $\varepsilon, \varepsilon' \in \{-1, 1, 4\}$  with  $|\varepsilon| \neq |\varepsilon'|$ . If  $\varepsilon = 4$ , then  $|\varepsilon_i| = 1$  for all  $i$ , contradicting  $|\varepsilon'| = 1$ . Similarly,  $\varepsilon' = 4$  is impossible. Thus, we must have  $\varepsilon^2 = \varepsilon'^2$  and hence,  $\varepsilon\alpha^2 \equiv \pm \varepsilon \pmod{n}$ . This implies  $\varepsilon = 0$  or  $\varepsilon = 3$ . If  $\varepsilon = 3$ , then  $\varepsilon_0\varepsilon_2 \neq 0$  and hence,  $\varepsilon'$  is even, contradicting the assumption  $\varepsilon' = \pm \varepsilon$ . Hence,  $\varepsilon = \varepsilon' = 0$ . Now, if none of the  $\varepsilon_i$  is zero, then (6)

implies that a subpath of the 8-cycle is a coiled 4-cycle, which is impossible. It is now easy to see that we must have  $\varepsilon_i = 0$  for all  $i$  and the conclusion follows.  $\square$

We would now like to classify the non-coiled 8-cycles in  $M(\alpha; 4, n)$ . It is not difficult to check that the following are the only possible block sequences for a non-coiled 8-cycle:

- (a)  $\langle\langle i, i+1, i+2, i+1, i, i+1, i+2, i+1 \rangle\rangle$ ,
- (b)  $\langle\langle i, i+1, i+2, i+3, i+2, i+1, i, i+1 \rangle\rangle$ ,
- (c)  $\langle\langle i, i+1, i+2, i+1, i, i+1, i+2, i+3 \rangle\rangle$ ,
- (d)  $\langle\langle i, i+1, i+2, i+1, i, i+1, i, i+1 \rangle\rangle$ ,
- (e)  $\langle\langle i, i+1, i, i+1, i+2, i+3, i+2, i+3 \rangle\rangle$ ,
- (f)  $\langle\langle i, i+1, i, i+1, i, i+1, i+2, i+3 \rangle\rangle$ ,
- (g)  $\langle\langle i, i+1, i+2, i+3, i, i+3, i+2, i+1 \rangle\rangle$ ,
- (h)  $\langle\langle i, i+1, i+2, i+3, i+2, i+3, i+2, i+1 \rangle\rangle$ ,
- (i)  $\langle\langle i, i+1, i+2, i+3, i+2, i+1, i+2, i+1 \rangle\rangle$ ,
- (j)  $\langle\langle i, i+1, i+2, i+1, i+2, i+1, i+2, i+1 \rangle\rangle$ ,
- (k)  $\langle\langle i, i+1, i+2, i+1, i+2, i+1, i, i+1 \rangle\rangle$ ,
- (l)  $\langle\langle i, i+1, i, i+1, i, i+1, i, i+1 \rangle\rangle$  and
- (m)  $\langle\langle i, i+1, i, i+1, i+2, i+1, i+2, i+3 \rangle\rangle$ .

**Lemma 3.10.** *Let  $n$  be odd and let  $\alpha^4 \equiv -1 \pmod{n}$ . The following is the list of all type sequences of the non-coiled 8-cycles occurring in  $M(\alpha; 4, n)$ . Each type sequence is preceded by a letter corresponding to its block sequence.*

- $n \notin \{17, 41, 73, 97\}$ :
  - (a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$ ,
- $n = 17, \alpha = 2$ :
  - (a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$ ,
  - (b<sub>1</sub>)  $\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle$ ,
  - (c<sub>1</sub>)  $\langle 1, \alpha, \alpha, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle$ ,
  - (c<sub>2</sub>)  $\langle 1, \alpha, \alpha, -1, -1, \alpha, \alpha^2, \alpha^3 \rangle$ ,
- $n = 41, \alpha = 3$ :
  - (a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$ ,
  - (d)  $\langle 1, -\alpha, -\alpha, 1, 1, 1, 1, 1 \rangle$ ,
  - (e<sub>1</sub>)  $\langle 1, 1, 1, -\alpha, \alpha^2, \alpha^2, \alpha^2, -\alpha^3 \rangle$ ,
- $n = 73, \alpha = 10$ :
  - (a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$ ,
  - (f<sub>1</sub>)  $\langle 1, 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle$ ,

- $n = 97$ ,  $\alpha = 33$ :

$$(a) \langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle,$$

$$(f_2) \langle 1, 1, 1, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle.$$

**Proof.** A cycle with a given type sequence occurs in the graph  $M(\alpha; 4, n)$  if and only if the corresponding congruence equality holds. For each of the block sequences (a)–(m) we find all possible non-equivalent type sequences associated with it. To the congruence equation of each of the type sequences we apply the squaring elimination method described in the proof of Lemma 3.8 to determine whether the type sequence occurs in the graph or not. Throughout this proof let  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$ .

The type sequences with the block sequence (a) are

$$(a_1) \langle 1, \delta_1 \alpha, \delta_1 \alpha, -1, -1, -\delta_1 \alpha, -\delta_1 \alpha, 1 \rangle,$$

$$(a_2) \langle 1, \delta_1 \alpha, \delta_1 \alpha, 1, 1, \delta_1 \alpha, \delta_1 \alpha, 1 \rangle,$$

$$(a_3) \langle 1, \delta_1 \alpha, \delta_1 \alpha, 1, 1, -\delta_1 \alpha, -\delta_1 \alpha, 1 \rangle \text{ and}$$

$$(a_4) \langle 1, \delta_1 \alpha, \delta_1 \alpha, -1, -1, \delta_1 \alpha, \delta_1 \alpha, 1 \rangle.$$

The type sequences  $(a_2), (a_3)$  and  $(a_4)$  imply  $4 + 4\delta_1 \alpha \equiv 0 \pmod{n}$ ,  $4 \equiv 0 \pmod{n}$ , and  $4\delta_1 \alpha \equiv 0 \pmod{n}$ , respectively, and each of these congruences is easily seen to yield a contradiction. The congruence equation for  $(a_1)$ , however, holds for any  $n$  and any  $\alpha$ . We obtain  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$  and  $\langle 1, -\alpha, -\alpha, -1, -1, \alpha, \alpha, 1 \rangle$ , but these two type sequences are equivalent and hence it is enough to make a note of

$$\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle.$$

The type sequences with the block sequence (b) are

$$(b_1) \langle 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_2 \alpha^2, \delta_1 \alpha, 1, 1, 1 \rangle \text{ and}$$

$$(b_2) \langle 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_2 \alpha^2, -\delta_1 \alpha, 1, 1, 1 \rangle.$$

The type  $(b_1)$  gives

$$4 + 2\delta_1 \alpha + 2\delta_2 \alpha^2 \equiv 0 \pmod{n}, \quad (7)$$

which (after applying the squaring elimination method) implies  $n \in \{9, 17\}$ . But  $\varphi(9) = 6$  is not divisible by 8 so that  $n = 17$ . Using  $\alpha = 2$  (by Lemma 3.7 we may pick any order-eight element of  $Z_{17}^*$ ) we can check that Eq. (7) is satisfied only for  $\delta_1 = 1$ ,  $\delta_2 = -1$ , and this produces the type sequence

$$\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle.$$

From  $(b_2)$  we get  $4 + 2\delta_2 \alpha^2 \equiv 0 \pmod{n}$ , which yields  $n = 5$ , a contradiction.

The block sequence (c) is associated with one of the following type sequences:

$$(c_1) \langle 1, \delta_1 \alpha, \delta_1 \alpha, 1, 1, -\delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3 \rangle,$$

$$(c_2) \langle 1, \delta_1 \alpha, \delta_1 \alpha, -1, -1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3 \rangle,$$

$$(c_3) \langle 1, \delta_1 \alpha, \delta_1 \alpha, -1, -1, -\delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3 \rangle \text{ or}$$

$$(c_4) \langle 1, \delta_1 \alpha, \delta_1 \alpha, 1, 1, \delta_1 \alpha, \delta_2 \alpha^2, \delta_3 \alpha^3 \rangle.$$

The equation for  $(c_1)$  is

$$3 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n} \quad (8)$$

and this implies  $n = 17$ . Using  $\alpha = 2$  we can see that (8) can be satisfied only when  $\delta_1 = \delta_2 = \delta_3 = 1$  and thus, we obtain the type sequence

$$\langle 1, \alpha, \alpha, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle.$$

The equation for  $(c_2)$  is

$$-1 + 3\delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n} \quad (9)$$

and again this implies  $n = 17$ . Using  $\alpha = 2$ , (9) holds only for  $\delta_1 = \delta_2 = \delta_3 = 1$ . Thus, we get

$$\langle 1, \alpha, \alpha, -1, -1, \alpha, \alpha^2, \alpha^3 \rangle.$$

The equation for  $(c_3)$  is  $-1 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$ . Since the graph has coiled girth 8, it can not be satisfied.

The equation for  $(c_4)$  is  $3 + 3\delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$  and this implies  $n \in \{25, 49\}$ , which is impossible since neither  $\varphi(25) = 20$  nor  $\varphi(49) = 42$  is divisible by 8.

The block sequence  $(d)$  is associated with the type sequence

$$\langle 1, \delta_1\alpha, \delta_1\alpha, 1, 1, 1, 1, 1 \rangle.$$

The equation is  $6 + 2\delta_1\alpha \equiv 0 \pmod{n}$ , which implies  $n = 41$ . Using  $\alpha = 3$ ,  $\delta_1 = -1$  is forced and so we have the type sequence

$$\langle 1, -\alpha, -\alpha, 1, 1, 1, 1, 1 \rangle.$$

With the block sequence  $(e)$  we are looking for the type sequence

$$\langle 1, 1, 1, \delta_1\alpha, \delta_2\alpha^2, \delta_2\alpha^2, \delta_2\alpha^2, \delta_3\alpha^3 \rangle,$$

which requires

$$3 + \delta_1\alpha + 3\delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}.$$

By squaring elimination we need  $n = 41$ . Using  $\alpha = 3$ , we can check that there are two possibilities: either  $\delta_1 = \delta_3 = -1$  and  $\delta_2 = 1$ , or  $\delta_1 = \delta_2 = -1$  and  $\delta_3 = 1$ . However, these two possibilities yield the same type sequence, that is,

$$\langle 1, 1, 1, -\alpha, \alpha^2, \alpha^2, \alpha^2, -\alpha^3 \rangle.$$

The block sequence  $(f)$  requires the type sequence

$$\langle 1, 1, 1, 1, 1, \delta_1\alpha, \delta_2\alpha^2, \delta_3\alpha^3 \rangle$$

and so the equation is

$$5 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}. \quad (10)$$

This implies  $n \in \{73, 97\}$ .

Using  $\alpha = 10$  for  $n = 73$ , (10) holds only for  $\delta_1 = -1$  and  $\delta_2 = \delta_3 = 1$ . We obtain the type sequence

$$\langle 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle.$$

For  $n = 97$  and  $\alpha = 33$ , (10) can be satisfied only for  $\delta_1 = \delta_2 = \delta_3 = -1$  giving the type sequence

$$\langle 1, 1, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle.$$

The congruence equalities of the remaining block sequences (g)–(m) can be examined in a way similar to the above to show that they cannot be satisfied.  $\square$

We continue with an analogue of the previous lemma for the case that the order of  $\alpha$  is 4.

**Lemma 3.11.** *Let  $n$  be odd and let  $\alpha$  have order 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$ . If  $n$  is not a multiple of 5, the following is the list of all type sequences of the non-coiled 8-cycles occurring in  $M = M(\alpha; 4, n)$ . Each type sequence is preceded by a letter corresponding to its block sequence.*

•  *$M$  has coiled girth 8:*

(a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle,$

•  *$M$  has coiled girth 4:*

(a)  $\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle,$

(c<sub>3</sub>)  $\langle 1, -\alpha, -\alpha, -1, -1, \alpha, -\alpha^2, -\alpha^3 \rangle.$

*If  $n$  is a multiple of 5 other than 15, at most two of the following type sequences occur in addition to the type sequences above:*

(b<sub>11</sub>)  $\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle,$

(b<sub>12</sub>)  $\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha, 1, 1, 1 \rangle,$

(e<sub>2</sub>)  $\langle 1, 1, 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, \alpha^3 \rangle,$

(e<sub>3</sub>)  $\langle 1, 1, 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^3 \rangle,$

(h<sub>1</sub>)  $\langle 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, \alpha, 1 \rangle,$  and

(h<sub>2</sub>)  $\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha, 1 \rangle.$

*If  $n = 15$  and  $\alpha = 2$ , then the coiled girth is 4 and the additional type sequences are:*

(b<sub>11</sub>)  $\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle,$

(c<sub>4</sub>)  $\langle 1, -\alpha, -\alpha, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle,$

(e<sub>3</sub>)  $\langle 1, 1, 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^3 \rangle,$

(f<sub>1</sub>)  $\langle 1, 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle,$  and

(m)  $\langle 1, 1, 1, -\alpha, -\alpha, -\alpha, -\alpha^2, -\alpha^3 \rangle.$

**Proof.** We shall examine the type sequences associated with the block sequences (a)–(m) in a way very similar to that of the proof of Lemma 3.10, except that when applying the squaring elimination method we are now using  $(\alpha^2)^2 \equiv 1 \pmod{n}$ . We also let  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$  and we label the type sequences as before. Since  $\alpha^2 \not\equiv -1 \pmod{n}$ , note that  $n$  is not a prime power.

The case with the block sequence (a) is done in exactly the same way as in the proof of Lemma 3.10.

The type sequence  $(b_1)$  requires  $4 + 2\delta_1\alpha + 2\delta_2\alpha^2 \equiv 0 \pmod{n}$ . This implies  $5\alpha^2 \equiv 5 \pmod{n}$ , that is,  $n \equiv 0 \pmod{5}$ . Consequently,  $\delta_2 = -1$ , and we get two possible type sequences,

$$\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle$$

and

$$\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha, 1, 1, 1 \rangle.$$

When  $n = 15$ , we may assume that  $\alpha = 2$ , and only the first type sequence occurs.

It is not difficult to see that the equations for the type sequences  $(b_2)$ ,  $(c_1)$ , and  $(c_2)$  cannot be satisfied.

The type sequence  $(c_3)$  is  $\langle 1, \delta_1\alpha, \delta_1\alpha, -1, -1, -\delta_1\alpha, \delta_2\alpha^2, \delta_3\alpha^3 \rangle$  and the corresponding equation is  $-1 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$ . This equation can be satisfied if and only if  $M$  has coiled girth 4. In that case we may assume by Lemma 3.9 that  $\delta_1 = \delta_2 = \delta_3 = -1$ . Hence,  $M$  contains 8-cycles with a type sequence

$$\langle 1, -\alpha, -\alpha, -1, -1, \alpha, -\alpha^2, -\alpha^3 \rangle.$$

The equation for the type sequence  $(c_4)$  is  $3 + 3\delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$ . This implies  $n = 15$ . Using  $\alpha = 2$ , the equation is satisfied only for  $\delta_1 = \delta_2 = \delta_3 = -1$ . We obtain the type sequence

$$\langle 1, -\alpha, -\alpha, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle.$$

With the block sequence (e), the equation  $3 + \delta_1\alpha + 3\delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$  is needed. This implies  $\delta_2 = \delta_1\delta_3 = -1$  and  $5\alpha^2 \equiv 5 \pmod{n}$  so that  $n \equiv 0 \pmod{5}$ . We get two possible type sequences,

$$\langle 1, 1, 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, \alpha^3 \rangle$$

and

$$\langle 1, 1, 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^3 \rangle.$$

For  $n = 15$  and  $\alpha = 2$ , the equation of the second type sequence is satisfied.

The block sequence (f) requires  $5 + \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \equiv 0 \pmod{n}$ . This implies  $n \in \{15, 45\}$ . For  $n = 15$  and  $\alpha = 2$ , we obtain the type sequence

$$\langle 1, 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle.$$

When  $n = 45$ , we may assume that  $\alpha = 8$ , but the equation cannot be satisfied.

The block sequence (h) is associated with one of  $2 + 4\delta_2\alpha^2 \equiv 0 \pmod{n}$  and  $2 + 2\delta_1\alpha + 4\delta_2\alpha^2 \equiv 0 \pmod{n}$ . The first equation cannot be satisfied. The second equation implies  $\delta_2 = -1$  and  $5\alpha^2 \equiv 5 \pmod{n}$  so that  $n \equiv 0 \pmod{5}$ . We get two possible type sequences,

$$\langle 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, \alpha, 1 \rangle$$

and

$$\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha, 1 \rangle.$$

None of these types occurs when  $n = 15$  and  $\alpha = 2$ .

The equations for the block sequences (d), (g), (i)–(l) are easily seen to be impossible to satisfy.

The equation for the block sequence (m) is the same as for the type sequence (c<sub>4</sub>) and is therefore satisfied only for  $n = 15$ ,  $\alpha = 2$ , and  $\delta_1 = \delta_2 = \delta_3 = -1$ . We obtain the type sequence

$$\langle 1, 1, 1 - \alpha, -\alpha, -\alpha, -\alpha^2, -\alpha^3 \rangle.$$

Now let  $n$  be a multiple of 5 other than 15. First notice that no pair of type sequences with the same block sequence among (b<sub>11</sub>), (b<sub>12</sub>), (e<sub>2</sub>), (e<sub>3</sub>), (h<sub>1</sub>) and (h<sub>2</sub>) can occur together. Subtracting the equation  $2 + 2\alpha - 4\alpha^2 \equiv 0 \pmod{n}$  for the type (h<sub>1</sub>) from the equation  $4 + 2\alpha - 2\alpha^2 \equiv 0 \pmod{n}$  for the type (b<sub>11</sub>) we obtain  $\alpha^2 \equiv -1 \pmod{n}$ , a contradiction. Similarly, we show that the types (b<sub>12</sub>) and (h<sub>2</sub>) cannot occur together. Adding the equations  $4 + 2\alpha - 2\alpha^2 \equiv 0 \pmod{n}$  and  $2 - 2\alpha - 4\alpha^2 \equiv 0 \pmod{n}$  for the types (b<sub>11</sub>) and (h<sub>2</sub>), respectively, we obtain  $3\alpha^2 \equiv 3 \pmod{n}$  which, together with  $5\alpha^2 \equiv 5 \pmod{n}$ , implies  $\alpha^2 \equiv 1 \pmod{n}$ , a contradiction. Similarly, we show that the types (b<sub>12</sub>) and (h<sub>1</sub>) cannot occur together. Hence, at most two of the type sequences (b<sub>11</sub>), (b<sub>12</sub>), (e<sub>2</sub>), (e<sub>3</sub>), (h<sub>1</sub>), and (h<sub>2</sub>) appear in the graph.  $\square$

In the next two lemmas we determine the 2-path codes of the graphs  $M(\alpha; 4, n)$  with  $n$  odd and  $\alpha^2 \not\equiv -1 \pmod{n}$  by listing the 2-path codes of the type sequences from Lemmas 3.8–3.11.

**Lemma 3.12.** *Let  $n$  be odd and let  $\alpha^4 \equiv -1 \pmod{n}$ . If  $n \notin \{17, 41, 73, 97\}$ , then the 2-path code of  $M(\alpha; 4, n)$  is (2, 5, 5). The 2-path codes of the graphs  $M(2; 4, 17)$ ,  $M(3; 4, 41)$ ,  $M(10; 4, 73)$ , and  $M(33; 4, 97)$ , are (8, 12, 12), (9, 9, 6), (6, 8, 6) and (6, 6, 8), respectively.*

**Proof.** First notice that it is enough to count the number of 8-cycles for the 2-paths with jump sequences  $\langle 1, 1 \rangle$ ,  $\langle \alpha^3, 1 \rangle$  and  $\langle -\alpha^3, 1 \rangle$ .

By Lemma 3.8, a 2-path with the jump sequence  $\langle \alpha^3, 1 \rangle$  or  $\langle -\alpha^3, 1 \rangle$  lies in exactly four coiled 8-cycles, and a 2-path with the jump sequence  $\langle 1, 1 \rangle$  does not lie in a coiled 8-cycle. Thus, it remains to consider the non-coiled 8-cycles. By Lemma 3.10, the only

Table 1

	Type sequence	2-Path code
(a)	$\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$	(2, 1, 1)
(b <sub>1</sub> )	$\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle$	(2, 1, 1)
(c <sub>1</sub> )	$\langle 1, \alpha, \alpha, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle$	(2, 3, 3)
(c <sub>2</sub> )	$\langle 1, \alpha, \alpha, -1, -1, \alpha, \alpha^2, \alpha^3 \rangle$	(2, 3, 3)
(d)	$\langle 1, -\alpha, -\alpha, 1, 1, 1, 1, 1 \rangle$	(3, 1, 0)
(e <sub>1</sub> )	$\langle 1, 1, 1, -\alpha, \alpha^2, \alpha^2, \alpha^2, -\alpha^3 \rangle$	(4, 3, 1)
(f <sub>1</sub> )	$\langle 1, 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle$	(4, 3, 1)
(f <sub>2</sub> )	$\langle 1, 1, 1, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle$	(4, 1, 3)
Coiled		(0, 4, 4)

possible type sequences for a non-coiled 8-cycle are those denoted by (a), (b<sub>1</sub>), (c<sub>1</sub>), (c<sub>2</sub>), (d), (e<sub>1</sub>), (e<sub>2</sub>), (f<sub>1</sub>) and (f<sub>2</sub>). For each type sequence, all possible jump sequences are obtained by multiplying each term of the type sequence by  $\alpha^i$ ,  $i = 0, 1, \dots, 7$ . Here, one has to be careful, because two distinct powers of  $\alpha$  might yield the same jump sequence. (This happens if the sequence has some kind of symmetry.) Once we have found all possible pairwise distinct jump sequences of a given type sequence, we simply count the number of occurrences of the jump sequence of a 2-path. Note that  $\langle -1, -1 \rangle$ ,  $\langle -1, -\alpha^3 \rangle$  and  $\langle -1, \alpha^3 \rangle$  count as occurrences of  $\langle 1, 1 \rangle$ ,  $\langle \alpha^3, 1 \rangle$  and  $\langle -\alpha^3, 1 \rangle$ , respectively. With jump sequence  $\langle 1, 1 \rangle$  we have to distinguish between occurrences of  $\langle 1, 1 \rangle$  with the block sequence  $\langle\langle i+1, i, i+1 \rangle\rangle$  and those with corresponding block sequence  $\langle\langle i, i+1, i \rangle\rangle$ . But since the number of occurrences of the subsequences of the form  $\langle\langle i+1, i, i+1 \rangle\rangle$  in a block sequence of any cycle is the same as the number of occurrences of the subsequences of the form  $\langle\langle i, i+1, i \rangle\rangle$ , we can limit ourselves to counting the number of occurrences of the jump sequence  $\langle 1, 1 \rangle$  with corresponding block sequence of the form  $\langle\langle i+1, i, i+1 \rangle\rangle$ .

Following the above method for every type sequence we obtain Table 1.

Using Lemma 3.10, we now sum up the 2-path codes of the type sequences corresponding to the given value of  $n$  to obtain the 2-path code of the graph  $M(\alpha; 4, n)$ . This completes the proof.  $\square$

**Lemma 3.13.** *Let  $n$  be odd and let  $\alpha$  have order 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$ . When  $n$  is not a multiple of 5, the 2-path code of  $M = M(\alpha; 4, n)$  is (4, 7, 8) if  $M$  has coiled girth 4, and (2, 5, 5) if  $M$  has coiled girth 8. When  $n$  is a multiple of 5 other than 15, the 2-path code of  $M$  is either (4, 7, 8), (6, 8, 9), or (8, 9, 10) if  $M$  has coiled girth 4, and either (2, 5, 5), (4, 6, 6), or (6, 7, 7) if  $M$  has coiled girth 8. The 2-path code of  $M(2; 4, 15)$  is (18, 15, 18).*

**Proof.** We use the method described in the proof of the preceding lemma and the information about the 8-cycles from Lemmas 3.9 and 3.11. The only difference is that, since  $-1 \notin \{\alpha^i : i = 0, 1, 2, 3\}$ , all possible jump sequences are obtained from a given type sequence by multiplying each term of the type sequence by  $\alpha^i$  and  $-\alpha^i$ ,  $i = 0, 1, 2, 3$ .



Table 2

	Type sequence	2-Path code
(a)	$\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$	(2, 1, 1)
Coiled		(0, 4, 4)

Table 3

	Type sequence	2-Path code
(a)	$\langle 1, \alpha, \alpha, -1, -1, -\alpha, -\alpha, 1 \rangle$	(2, 1, 1)
(c <sub>3</sub> )	$\langle 1, -\alpha, -\alpha, -1, -1, \alpha, -\alpha^2, -\alpha^3 \rangle$	(2, 2, 4)
Coiled		(0, 4, 3)

Table 4

	Type sequence	2-Path code
(b <sub>11</sub> )	$\langle 1, \alpha, -\alpha^2, -\alpha^2, \alpha, 1, 1, 1 \rangle$	(2, 1, 1)
(b <sub>12</sub> )	$\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha, 1, 1, 1 \rangle$	(2, 1, 1)
(c <sub>4</sub> )	$\langle 1, -\alpha, -\alpha, 1, 1, -\alpha, -\alpha^2, -\alpha^3 \rangle$	(2, 2, 4)
(e <sub>2</sub> )	$\langle 1, 1, 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, \alpha^3 \rangle$	(2, 1, 1)
(e <sub>3</sub> )	$\langle 1, 1, 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^3 \rangle$	(2, 1, 1)
(f <sub>1</sub> )	$\langle 1, 1, 1, 1, 1, -\alpha, \alpha^2, \alpha^3 \rangle$	(4, 2, 2)
(h <sub>1</sub> )	$\langle 1, \alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, \alpha, 1 \rangle$	(2, 1, 1)
(h <sub>2</sub> )	$\langle 1, -\alpha, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha^2, -\alpha, 1 \rangle$	(2, 1, 1)
(m)	$\langle 1, 1, 1, -\alpha, -\alpha, -\alpha, -\alpha^2, -\alpha^3 \rangle$	(4, 2, 2)

Again we have to make sure that we count occurrences of the jump sequences  $\langle 1, 1 \rangle$ ,  $\langle \alpha^3, 1 \rangle$  and  $\langle -\alpha^3, 1 \rangle$  only in distinct jump sequences of the 8-cycles.

If  $M$  has coiled girth 8, we obtain following Table 2.

If  $M$  has coiled girth 4, the table (Table 3) is somewhat different.

The 2-path codes of the additional type sequences for the case that  $n$  is a multiple of 5 are given in Table 4.

Using the information on the occurrence of various type sequences in the graphs and summing up the 2-path codes of the appropriate type sequences of the cycles we obtain the result.  $\square$

We now have all the information we need to prove that certain graphs  $M(\alpha; 4, n)$  are  $\frac{1}{2}$ -transitive.

**Proposition 3.14.** *Let  $n$  be odd. In addition, either let the order of  $\alpha$  be 8 with  $\alpha^4 \equiv -1 \pmod{n}$  or let the order of  $\alpha$  be 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$ . Let  $(k_1, k_2, k_3)$  be the 2-path code of the graph  $M = M(\alpha; 4, n)$ . If  $k_2 \neq k_1 \neq k_3$ , then  $M$  is  $\frac{1}{2}$ -transitive.*

**Proof.** First, we show that the blocks of  $M$  are in fact the blocks of imprimitivity of its automorphism group. Let  $V^i$  be a block of  $M$  and let  $\sigma \in \text{Aut}(M)$  be such that

$V^i \cap \sigma(V^i) \neq \emptyset$ . Let  $u$  and  $v$  be vertices in  $V^i$  such that  $\sigma(u) = v$ . Since  $k_1 \neq k_2, k_3$  the set of 2-paths with the type sequence  $\langle 1, 1 \rangle$  is fixed setwise by  $\sigma$ . Hence,  $\sigma$  maps the set  $N^2(u) \cap V^i$  to the set  $N^2(v) \cap V^i$ , where  $N^2(u)$  denotes the set of all vertices at distance 2 from  $u$ . Consequently,  $\sigma$  takes all vertices in  $V^i$  at an even distance from  $u$  to vertices in  $V^i$ . Since  $n$  is odd we have  $\sigma(V^i) = V^i$ .

Now suppose that  $M$  is arc-transitive. Then there exists an automorphism  $\theta$  that interchanges the vertices  $v_0^0$  and  $v_1^1$ . Thus,  $\theta$  interchanges the blocks  $V^0$  and  $V^1$ , and the blocks  $V^2$  and  $V^3$ . This implies that any coiled 8-cycle containing the edge  $v_0^0 v_1^1$  is mapped to a coiled 8-cycle containing the edge  $v_0^0 v_1^1$ . Examining all coiled 8-cycles containing the edge  $v_0^0 v_1^1$  we can see that the vertex  $v_{1+\alpha+\alpha^2-\alpha^3}^0$  from the coiled 8-cycle

$$C = v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3 v_{1+\alpha+\alpha^2-\alpha^3}^0 v_{\alpha+\alpha^2-\alpha^3}^1 v_{\alpha^2-\alpha^3}^2 v_{-\alpha^3}^3 v_0^0$$

must be mapped to the vertex  $v_{\delta_1\alpha+\delta_2\alpha^2+\delta_3\alpha^3}^1$  for some  $\delta_1, \delta_2, \delta_3 \in \{-1, 1\}$ . Since the action of  $\theta$  on  $V^0$  is given by  $\theta(v_j^0) = v_{-j+1}^1$ , this forces

$$1 - (1 + \alpha + \alpha^2 - \alpha^3) \equiv \delta_1\alpha + \delta_2\alpha^2 + \delta_3\alpha^3 \pmod{n},$$

which by the proofs of Lemmas 3.8 and 3.9 implies  $\delta_1 = \delta_2 = -1$ ,  $\delta_3 = 1$ . Thus,  $C$  is mapped to the coiled 8-cycle

$$v_0^0 v_1^1 v_{1-\alpha}^2 v_{1-\alpha-\alpha^2}^3 v_{1-\alpha-\alpha^2+\alpha^3}^0 v_{-\alpha-\alpha^2+\alpha^3}^1 v_{-\alpha^2+\alpha^3}^2 v_{\alpha^3}^3 v_0^0$$

and so  $\theta(v_{1+\alpha}^2) = v_{\alpha^3}^3$ . Similarly, the coiled 8-cycle

$$v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha-\alpha^2}^3 v_{1+\alpha-\alpha^2+\alpha^3}^0 v_{\alpha-\alpha^2+\alpha^3}^1 v_{-\alpha^2+\alpha^3}^2 v_{\alpha^3}^3 v_0^0$$

must be mapped to

$$v_0^0 v_1^1 v_{1-\alpha}^2 v_{1-\alpha+\alpha^2}^3 v_{1-\alpha+\alpha^2-\alpha^3}^0 v_{-\alpha+\alpha^2-\alpha^3}^1 v_{\alpha^2-\alpha^3}^2 v_{-\alpha^3}^3 v_0^0$$

and so  $\theta(v_{1+\alpha}^2) = v_{-\alpha^3}^3$ , which is a contradiction.

This proves that  $M(\alpha; 4, n)$  cannot be arc-transitive.  $\square$

The next lemma captures the exceptional cases to which the previous proposition does not apply.

**Lemma 3.15.** *The graphs  $M(3; 4, 41)$ ,  $M(10; 4, 73)$ ,  $M(33; 4, 97)$ , and  $M(2; 4, 15)$  are  $\frac{1}{2}$ -transitive.*

**Proof.** Let  $M$  be one of the graphs  $M(3; 4, 41)$ ,  $M(10; 4, 73)$ ,  $M(33; 4, 97)$ , and  $M(2; 4, 15)$ . We will show that any automorphism  $\theta$  of  $M$  that interchanges the vertices  $v_0^0$  and  $v_1^1$  interchanges the blocks  $V^0$  and  $V^1$ . The proof can then be completed exactly as in the previous proposition.

Note that in the graphs  $M(3; 4, 41)$ ,  $M(33; 4, 97)$  and  $M(2; 4, 15)$  a 4-path with the type sequence  $\langle 1, 1, \alpha, \alpha^2 \rangle$  cannot be mapped to a 4-path with the type sequence

$\langle 1, -\alpha, -\alpha^2, -\alpha^3 \rangle$  or  $\langle 1, \alpha, \alpha^2, -\alpha^3 \rangle$ , and a 4-path with the type sequence  $\langle 1, \alpha, \alpha^2, \alpha^2 \rangle$  cannot be mapped to a 4-path with the type sequence  $\langle 1, -\alpha, -\alpha^2, -\alpha^3 \rangle$ . This can be seen by counting the number of 8-cycles containing a 4-path with the given type sequence. Similarly, observe that in the graph  $M(10; 4, 73)$  a 4-path with the type sequence  $\langle 1, -\alpha, \alpha^2, \alpha^2 \rangle$  cannot be mapped to a 4-path with the type sequence  $\langle 1, -\alpha, \alpha^2, \alpha^3 \rangle$  or  $\langle 1, \alpha, -\alpha^2, \alpha^3 \rangle$ , and a 4-path with the type sequence  $\langle 1, 1, -\alpha, \alpha^2 \rangle$  cannot be mapped to a 4-path with the type sequence  $\langle 1, -\alpha, \alpha^2, \alpha^3 \rangle$ .

First, let  $M$  be one of the graphs  $M(3; 4, 41)$  and  $M(33; 4, 97)$ . Let  $\sigma$  be an automorphism of  $M$  that fixes the vertices  $v_0^0$  and  $v_1^1$ . From the 2-path code of  $M$  it follows that  $\sigma$  fixes the vertices of the coiled 8-cycle

$$v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3 v_{1+\alpha+\alpha^2+\alpha^3}^0 v_{\alpha+\alpha^2+\alpha^3}^1 v_{\alpha^2+\alpha^3}^2 v_{\alpha^3}^3 v_0^0.$$

Since, by the above observation,  $\sigma$  cannot map the 4-path  $v_{-1}^1 v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3$  to the 4-path  $v_{-\alpha}^3 v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3$ ,  $\sigma$  also fixes the remaining neighbours  $v_{-1}^1$  and  $v_{-\alpha}^3$  of  $v_0^0$ . Since  $\sigma$  fixes  $v_{-1}^1$  and  $v_0^0$ , it fixes the vertices of the coiled 8-cycle

$$v_0^0 v_{-1}^1 v_{-1-\alpha}^2 v_{-1-\alpha-\alpha^2}^3 v_{-1-\alpha-\alpha^2-\alpha^3}^0 v_{-\alpha-\alpha^2-\alpha^3}^1 v_{-\alpha^2-\alpha^3}^2 v_{-\alpha^3}^3 v_0^0,$$

and since it cannot map the 4-path  $v_{-2}^0 v_{-1}^1 v_{-1-\alpha}^2 v_{-1-\alpha-\alpha^2}^3 v_{-1-\alpha-\alpha^2-\alpha^3}^0$  to the 4-path  $v_{-1+\alpha}^2 v_{-1}^1 v_{-1-\alpha}^2 v_{-1-\alpha-\alpha^2}^3 v_{-1-\alpha-\alpha^2-\alpha^3}^0$ , it also fixes  $v_{-2}^0$  and  $v_{-1+\alpha}^2$ . Continuing in this fashion along the  $2n$ -cycle  $M[V^0, V^1]$  we can see that  $\sigma$  fixes  $V^0$  and  $V^1$  pointwise. Hence,  $\sigma$  is the identity.

Now, suppose there exists  $\theta \in \text{Aut}(M)$  that reverses the edge  $v_0^0 v_1^1$ . Then  $\theta^2 = 1$  and  $\theta$  fixes the coiled 8-cycle

$$v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3 v_{1+\alpha+\alpha^2+\alpha^3}^0 v_{\alpha+\alpha^2+\alpha^3}^1 v_{\alpha^2+\alpha^3}^2 v_{\alpha^3}^3 v_0^0$$

interchanging pairs of its vertices in the obvious way. Consequently,  $\theta$  interchanges the sets  $\{v_{-1}^1, v_{-\alpha}^3\}$  and  $\{v_2^0, v_{1-\alpha}^2\}$ . However, since the 4-path  $v_{-1}^1 v_0^0 v_1^1 v_{1+\alpha}^2 v_{1+\alpha+\alpha^2}^3$  cannot be mapped to the 4-path  $v_{1-\alpha}^2 v_1^1 v_0^0 v_{\alpha^3}^3 v_{\alpha^2+\alpha^3}^2$ ,  $\theta$  in fact interchanges the vertices  $v_{-1}^1$  and  $v_2^0$ , as well as  $v_{-\alpha}^3$  and  $v_{1-\alpha}^2$ . Consequently,  $\theta$  maps the coiled 8-cycle

$$v_0^0 v_{-1}^1 v_{-1-\alpha}^2 v_{-1-\alpha-\alpha^2}^3 v_{-1-\alpha-\alpha^2-\alpha^3}^0 v_{-\alpha-\alpha^2-\alpha^3}^1 v_{-\alpha^2-\alpha^3}^2 v_{-\alpha^3}^3 v_0^0$$

to the coiled 8-cycle

$$v_1^1 v_2^0 v_{2-\alpha^3}^3 v_{2-\alpha^2-\alpha^3}^2 v_{2-\alpha-\alpha^2-\alpha^3}^1 v_{1-\alpha-\alpha^2-\alpha^3}^0 v_{1-\alpha-\alpha^2}^3 v_{1-\alpha}^2 v_1^1,$$

and interchanges the sets  $\{v_{-2}^0, v_{-1+\alpha}^2\}$  and  $\{v_3^1, v_{2+\alpha^3}^3\}$ . Again, since the 4-path  $v_{-2}^0 v_{-1}^1 v_{-1-\alpha}^2 v_{-1-\alpha-\alpha^2}^3 v_{-1-\alpha-\alpha^2-\alpha^3}^0$  cannot be mapped to the 4-path  $v_{2+\alpha^3}^3 v_2^0 v_{2-\alpha^3}^3 v_{2-\alpha^2-\alpha^3}^2 v_{2-\alpha-\alpha^2-\alpha^3}^1$ ,  $\theta$  in fact interchanges  $v_{-2}^0$  and  $v_3^1$ . Continuing this way along the  $2n$ -cycle  $M[V^0, V^1]$  we can see that  $\theta$  interchanges the blocks  $V^0$  and  $V^1$ .

The cases  $M = M(10; 4, 73)$  and  $M = M(2; 4, 15)$  can be settled in a similar way with the roles of the type sequences  $\langle \alpha^3, 1 \rangle$  and  $\langle -\alpha^3, 1 \rangle$  reversed. In the proof for  $M(2; 4, 15)$ , however, the coiled 8-cycles are substituted by coiled 4-cycles, and in

$M(10; 4, 73)$ , a 4-path is always ‘wrapped’ around the coiled 8-cycle in the direction opposite to the one chosen above.  $\square$

We are now ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** Lemma 3.2 takes care of the case  $\alpha^2 \equiv -1 \pmod{n}$ . If  $\alpha^4 \equiv -1 \pmod{n}$  and  $n \equiv 0 \pmod{4}$ , then  $M(\alpha; 4, n)$  does not exist by Lemma 3.5. If  $n \equiv 2 \pmod{4}$ , then, by Lemma 3.6,  $M(\alpha; 4, n)$  consists of two disjoint copies of  $M(\alpha \bmod (n/2); 4, (n/2))$ . Hence, we may assume that  $n$  is odd. If  $\alpha^4 \equiv -1 \pmod{n}$ ,  $M(\alpha; 4, n)$  is  $\frac{1}{2}$ -transitive by Lemmas 3.12 and 3.15, and by Proposition 3.14. If the order of  $\alpha$  is 4 with  $\alpha^2 \not\equiv -1 \pmod{n}$ ,  $M(\alpha; 4, n)$  is  $\frac{1}{2}$ -transitive by Lemmas 3.13 and 3.15, and by Proposition 3.14.  $\square$

Theorem 3.1 thus produces an infinite family of  $\frac{1}{2}$ -transitive graphs.

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